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Quantization of the Fractional Harmonic Oscillator in terms of Riesz Fractional Derivatives

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قدمت هذه الرسالة استكمالاً لمتطلبات الحصول على درجة الماجستير في الفيزياء في كلية العلوم في جامعة آل البيت.

Dedication

To my Mother

And to

My Wife and

Children

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Abstract

Fractional Lagrangians and fractional Hamiltonians for systems containing Riesz fractional derivatives (RFDs) have been formulated. The Hamilton's equations of motion are derived. Besides, the Hamilton-Jacobi formulations for these corresponding systems have been developed using the canonical method. The relevant Hamilton–Jacobi function has been obtained. An example for the fractional harmonic oscillator is discussed. The traditional results are recovered for integer-order derivatives.

The path integral quantization of the fractional harmonic oscillator using Riesz-Caputo fractional derivatives (RCFD) has been performed.

A Lagrangian for a damped Harmonic oscillator has been proposed in terms of Riesz's Fractional derivative, and the corresponding equation of motion has been obtained.

CHAPTER ONE

Introduction

1.1 Statement of the Problem

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders (including complex orders), and their applications in science. The seeds of fractional derivatives were planted over 300 years ago. It has been applied to almost every field of science; including Fluid Flow, Probability and Statistics, Control Theory of Dynamical Systems, Chemical Physics, Optics, Signal Processing, Electrical Engineering, Electrochemistry, Biology, Biophysics, Mechanics, Mechatronics and all branches of physics [Kilbas et al. 2006].

Various applications of fractional calculus are based on replacing the time derivative in an evolution equation with a derivative of fractional order. The results of several recent researches confirm that fractional derivatives seem to arise for mathematical reasons and many important results were reported [Magine 2004]. Classical Mechanics is one of the fields where fractional calculus generalized the classical calculus and it has many important applications, including classical and quantum mechanics, field theory, and optimal control.

During the past few years a special attention has been devoted to the fractional variations and their applications which gained importance in studying fractional mechanics and understanding constrained dynamics, both at the classical and quantum level. This is because, it is not suitable for all systems containing internal damping to be described properly within the classical picture (using only integer-order time derivatives) since the canonical coordinates of such systems do not remain linearly independent and certain constraints appear among them. For these reasons fractional studies of the properties of the Lagrangian and Hamiltonian formalisms are considered.

Fractional Lagrangians are constructed from classical Lagrangian by replacing the classical derivatives with chosen fractional derivatives and the fractional Euler–Lagrange equations are obtained as a result of a fractional variational' procedure [Agrawal 2002]. The first attempt to find the fractional Lagrangian and Hamiltonian for a given dissipative system is due to [Riewe 1996, 1997]. Important contributions were obtained in the field of variational principles by [Klimek 2001, 2002], [Agrawal 2006, 2007], [Rabei et al. 2004, 2007A], [Baleanu 2004] and [Baleanu and Agrawal 2006].

The fractional Hamilton–Jacobi formalism has been developed for quantizing constrained systems [Güler, 1992; Rabei, 1996; Rabei and Güler, 1992]. The equivalent Lagrangian method is used to obtain the set of Hamilton–Jacobi partial differential equations (HJPDEs) for constrained systems. A general solution of the set of HJPDEs of these systems has been proposed, so that the Hamilton–Jacobi function in configuration space has been obtained [Rabei et al. 2002].

The Euler–Lagrange equation and fractional Hamilton-Jacobi formulation for systems have been investigated for fractional discrete and continuous systems, mostly in terms of Riemann-Liouville (RL) and Caputo fractional derivatives [Rabei et al 2007] and [Rabei and Ababneh, 2008].

The conventional calculus of variations for systems containing Riesz fractional derivatives (RFD) has been extended by [Agrawal 2007] and [Baleanu 2007]. For integer α, the Riesz derivatives agree with traditional definitions, when α is 1. the right derivative is equal to the left derivative. This is not the case for Riemann and Caputo fractional derivatives; the right derivative is the negative of the left derivative. We have applied Riesz derivatives to fractional dynamics. This could give rise to opportunities in studying constrained systems, mainly because the Riesz derivative contains both the left and right derivatives. In addition, the fractional derivative of a function is given by a definite integrals, this depends on the values of the function over the entire interval. Therefore, fractional derivatives are suitable to model systems with long-range interactions in space and/or time.

In this thesis, the definition of a Riesz fractional potential is used to define RFD's. Two definitions are possible for RFD's: one analogous to the Riemann–Liouville fractional derivative (RLFD), and the other analogous to the Caputo fractional derivative (CFD). In this respect, using the Riesz derivative, we propose to generalize the notion of equivalent Lagrangians and Hamiltonians for the fractional cases. Moreover, the Riesz Caputo fractional derivative will be used to construct the fractional Hamilton-Jacobi equations for systems using the canonical method.

Quantization of the fractional harmonic oscillator FHO will be performed using a path-integral method. Moreover, a new Lagrangian formulation for a damped harmonic oscillator will be derived.

1.2 Motivation

The formulation of the fractional Hamilton-Jacobi equation using Riesz fractional derivatives is an interesting issue to be investigated, not only because nobody has used these derivatives before to build Hamilton-Jacobi equations, but also for the unique properties of Riesz derivatives mentioned above.

Another thing is to build a convincing fractional Lagrangian for a damped harmonic oscillator with RFD.

1.2 Synopsis of the Thesis

The plan of this thesis is as follows: In Chapter One, a brief introduction to the fractional calculus and previous studies is presented. In Chapter Two, some basic formulas of fractional derivatives are reviewed.

Chapter Three is divided to three sections: The first section contains a brief review of the fractional Lagrangian and fractional Hamiltonian approaches using Riesz fractional derivatives. The second one, fractional Hamilton-Jacobi formulation using Riesz fractional derivatives discusses. The third section, the fractional Harmonic Oscillator is solved using these formulation.

Chapter Four has two main sections: The first is an introduction to path integrals. The second section presents the quantization of the fractional Harmonic Oscillator within the path integral method.

Chapter Five contains the formulation a Lagrangian for the damped harmonic oscillator.

Finally, Chapter Six consist of a general summary and a partial list of open problems.

CHAPTER Two

Fractional Derivatives

Several definitions of a fractional derivative have been proposed, including the Riemann–Liouville, the Grunwald–Letnikov, the Weyl, the Caputo, the Marchaud, the Riesz, the Miller and Ross fractional derivatives. In this chapter, we shall present the basic definitions and properties of Riemann-Liouville, Caputo and Riesz fractional integrals and derivatives.

We begin with the left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ for a function $q\,(t)$ in the finite interval $[a, b]$ $(-\infty < a < b < \infty)$ on the real axis $\Re$.

The left Riemann-Liouville fractional (LRLF) integral reads

$$ {}_aI_t^{\alpha}\, q(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}\, q(\tau)d\tau. \qquad \alpha > 0 \qquad (2.1)$$

The right Riemann-Liouville fractional (RRLF) integral reads

$$ {}_tI_b^{\alpha}\, q(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1}\, q(\tau)d\tau, \qquad \alpha > 0 \qquad (2.2)$$

where $\Gamma(\alpha)$ denotes the Gamma function.

The next step is to define the Riesz potential as was given in [Agrawal 2007]:

$$ {}_a^RI_b^{\alpha} q(t) = \frac{1}{2\Gamma(\alpha)} \int_a^b |t-\tau|^{\alpha-1}\, q(\tau)d\tau. \qquad \alpha > 0 \qquad (2.3)$$

Equation (2.3) is called the Riesz potential or Riesz fractional integral, the limits going from $(-\infty \text{ to } \infty)$.

From Eqs, (2.1) to (2.3) we conclude that

$$ {}_{a}^{R}I_{b}^{\alpha} q(t) = \frac{1}{2}\left({}_{a}I_{t}^{\alpha}\, q(t) + {}_{t}I_{b}^{\alpha}\, q(t)\right). \qquad (2.4)$$

The left Riemann-Liouville fractional derivative (LRLFD) is defined as

$$ {}_{a}D_{t}^{\alpha}\, q(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n} \int_{a}^{t}(t-\tau)^{n-\alpha-1}\, q(\tau)d\tau = D^{n}\ {}_{a}I_{t}^{n-\alpha}\, q(t); \qquad (2.5)$$

while the right Riemann-Liouville fractional derivative (RRLFD) is defined as

$$ {}_{t}D_{b}^{\alpha}\, q(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^{n} \int_{t}^{b}(\tau-t)^{n-\alpha-1}\, q(\tau)d\tau = (-D)^{n}\ {}_{a}I_{t}^{n-\alpha}\, q(t). \qquad (2.6)$$

The fractional Riemann-Liouville derivatives have the following properties. The fractional derivative of a constant is not zero, namely;

$$ {}_{a}D_{t}^{\alpha}\, c = c\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}.$$

The RL derivative of a power of q has the following form:

$$ {}_{a}D_{t}^{\alpha}\, q^{\beta} = \frac{\Gamma(\alpha-1)\, q^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}. \qquad (2.7)$$

The Caputo derivative of fractional order of function $q\,(t)$ is defined as:

the left fractional derivative (LCFD):

$$ {}_{a}^{c}D_{t}^{\alpha} q(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(\tau-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^{n} q(\tau)d\tau = {}_{a}I_{t}^{n-\alpha} D^{n}\ q(t); \qquad (2.8)$$

the right fractional derivative (RCFD);

$$ {}_{t}^{C}D_{b}^{\alpha}\, q(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{b} (\tau - t)^{n-\alpha-1} \left(-\frac{d}{dt}\right)^{n} q(\tau) d\tau \;=\; {}_{t}I_{b}^{n-\alpha} (-D)^{n}\, q(t), \qquad (2.9) $$

where $n - 1 < \alpha < n \in \mathbb{Z}^{+}$.

In contrast with the RL derivative, the Caputo derivative of a constant is zero; and for a fractional differential equation, defined in terms of Caputo derivatives, standard boundary conditions are well defined.

For $0 < \alpha < 1$ we have the relation

$$ {}_{a}D_{t}^{\alpha}\, f(t) = {}_{a}^{C}D_{t}^{\alpha} f(t) - \frac{(t-a)^{-\alpha} f(a)}{\Gamma(1-\alpha)}, \qquad (2.10) $$

where D is the traditional derivative operator and α is the order of the derivative such that

$n - 1 < \alpha < n$. When α is an integer, the usual definition of a derivative is used. Note that for $\alpha = 1$, the left derivative is the negative of the right derivative.

Following the above analogy, we can represent the Riesz Riemann–Liouville fractional derivative, or simply the Riesz fractional derivative as (RFD),

$$ {}_{a}^{R}D_{b}^{\alpha}\, q(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{a}^{b} |t-\tau|^{n-\alpha-1} q(\tau) d\tau = D^{n}\; {}_{a}^{R}I_{t}^{n-\alpha} q(t). \qquad (2.11) $$

The Riesz-Caputo fractional derivative (RCFD) is defined as

$$ {}_{a}^{RC}D_{b}^{\alpha}\, q(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{b} |t-\tau|^{n-\alpha-1} \left(\frac{d}{dt}\right)^{n} q(\tau) d\tau = {}_{a}^{R}I_{t}^{n-\alpha} D^{n}\; q(t). \qquad (2.12) $$

Using the above equations, we conclude

$$ {}_{a}^{R}D_{b}^{\alpha} q(t) = \frac{1}{2}\Big({}_{a}D_{t}^{\alpha}\, q(t) + (-1)^{n}\;\; {}_{t}D_{b}^{\alpha}\, q(t)\Big); \qquad (2.13) $$

and

$$ {}^{RC}_{\ a}D^{\alpha}_{b} q(t) = \frac{1}{2}\left({}^{C}_{a}D^{\alpha}_{t}\, x(t) + (-1)^{n}\ {}^{C}_{t}D^{\alpha}_{b}\ q(t) \right). \qquad (2.14) $$

Unlike Riemann–Liouville and Caputo fractional derivatives when α is 1, the right derivative of the Riesz fractional derivative is equal to the left derivative. Thus, for integer α, the Riesz derivatives defined above agree with traditional definitions of a derivative.

From Eq. (2.13), the Riesz fractional derivative RFD of a constant is not zero, namely;

$$ {}^{R}_{a}D^{\alpha}_{b}\ c = \frac{1}{2} c \left(\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)} \right); $$

while from Eq. (2.14), the Riesz Caputo fractional derivative RCFD for a constant is zero.

Chapter Three

Fractional Hamilton- Jacobi Equations Using Riesz Derivatives

3.1 The Fractional Euler-Lagrange and Fractional Hamilton Equations with Riesz's Derivatives

Consider a fractional Lagrangian of the form $L_f = L(t, q, {}^{RC}_{a}D^{\alpha}_{b}q)$. The corresponding Euler– Lagrange equations are given as [Agrawal 2007]

$$\frac{\partial L_f}{\partial q} - {}^{R}_{a}D^{\alpha}_{b}\left(\frac{\partial L_f}{\partial\, {}^{RC}_{a}D^{\alpha}_{b}q}\right) = 0. \qquad (3.1.1)$$

Equation (3.1.1) represents the generalized Euler–Lagrange equation for the Fractional Calculus of Variation FCV problem defined in terms of the RCFD. We see that both RFD and RCFD automatically appear in the resulting Euler–Lagrange equation even when the Lagrangian function contains RCFD only [Agrawal 2007].

For a given fractional Lagrangian $L_f = L(t, q, {}^{RC}_{a}D^{\alpha}_{b}q)$, the fractional canonical momentum is defined as

$$p_{\alpha} = \frac{\partial L_f}{\partial\, {}^{RC}_{a}D^{\alpha}_{b}q} \ . \qquad (3.1.2)$$

Then, the corresponding fractional Hamiltonian (H_f) takes the form

$$H_f(q, p_{\alpha,}\ t) = p_\alpha\ ({}^{RC}_{a}D^{\alpha}_{b}q) - L_f\ . \qquad (3.1.3)$$

Taking the total differential of Eq. (3.1.3), we obtain

$$dH_f = p_\alpha d({}^{RC}_{a}D^{\alpha}_{b}q) + dp_\alpha\ {}^{RC}_{a}D^{\alpha}_{b}q - \frac{\partial L_f}{\partial q}dq - \frac{\partial L_f}{\partial\ {}^{RC}_{a}D^{\alpha}_{b}q}d({}^{RC}_{a}D^{\alpha}_{b}q) - \frac{\partial L_f}{\partial t}dt. \qquad (3.1.4)$$

Using again Eq. (3.1.1) and making use of Eq. (3.1.2), we get

$$dH_f = dp_\alpha\ {}^{RC}_{a}D^{\alpha}_{b}q - {}^{R}_{a}D^{\alpha}_{b}p_\alpha\ dq - \frac{\partial L_f}{\partial t}\ dt. \qquad (3.1.5)$$

This means that the Hamiltonian is a function of the form $H_f = H\ (q,p_\alpha,\ t)$. Thus, the total differential of this function takes the form

$$dH_f = \frac{\partial H_f}{\partial q}\ dq + \frac{\partial H_f}{\partial p_\alpha}\ dp_\alpha + \frac{\partial H_f}{\partial t}\ dt. \qquad (3.1.6)$$

Comparing Eq. (3.1.5) to Eq. (3.1.6), we get the fractional Hamilton's equations:

$$\frac{\partial H_f}{\partial p_\alpha} = {}^{RC}_{a}D^{\alpha}_{b}q, \qquad \frac{\partial H_f}{\partial q} = -\ {}^{R}_{a}D^{\alpha}_{b}p_\alpha, \qquad \frac{\partial H_f}{\partial t} = -\ \frac{\partial L_f}{\partial t} \qquad (3.1.7)$$

It is observed that the total time derivative of the fractional Hamiltonian can be written as

$$\frac{dH_f}{dt} = \dot{P}_\alpha\ {}^{RC}_{a}D^{\alpha}_{b}q - \dot{q}\ {}^{R}_{a}D^{\alpha}_{b}p_\alpha + \frac{\partial H_f}{\partial t}, \qquad (3.1.8)$$

which means that the fractional Hamiltonian is not a constant of motion, even though it is independent of time explicitly.

3.2 Fractional Hamilton-Jacobi Formulation with Riesz's Derivatives

The Hamilton–Jacobi Equation (HJE) is another formulation of classical mechanics and it is equivalent to other formulations such as Newton's laws of motion, Lagrangian mechanics and Hamiltonian mechanics. The Hamilton–Jacobi Equation is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely.

The Hamilton-Jacobi Equation is enormously useful in solving analytically and numerically equations of motion for classical particles. The main reason for its usefulness is that it yields all constants of motion automatically and the solution itself becomes formulated in terms of those constants of motion. It is also the only formulation of mechanics in which the motion of a particle can be represented as a wave. For this reason, the HJE is considered the "closest approach" of classical mechanics to quantum mechanics.

In this section, we formulate the Hamilton-Jacobi equation with Riesz fractional derivatives. In practice, the Hamilton–Jacobi technique becomes a useful computational tool only when a separation can be done. In general, coordinates q_i are said to be separable in the Hamilton–Jacobi equations when Hamilton's principal function can be split into two additive parts: one that depends only on the fractional derivatives of the generalized coordinates q; and another that is entirely independent of these derivatives. In the cases to which we shall apply the method of separation of variables, the Hamiltonian will be time-independent.

Now, let us consider the canonical transformations with the generating function $S({}^{RC}_{a}D^{\alpha-1}_{b}q, P_\alpha, t)$.

The new Hamiltonian H' will take the form

$$H'(Q,\mathbf{P}_\alpha, t) = \mathbf{P}_\alpha ({}^{RC}_{a}D^{\alpha}_{b}Q) - L'(Q, {}^{RC}_{a}D^{\alpha}_{b}Q, t), \qquad (3.2.1)$$

where Q, $\mathbf{P}$ are the new canonical coordinates and L' is the new Lagrangian.

To get the relation between the old Hamiltonian H and the new one H', both must obey Hamilton's principle:

$$\delta \int_{t1}^{t2} (p_\alpha \, {}^{RC}_{a}D^{\alpha}_{b}q - H) \, dt = 0; \quad \delta \int_{t1}^{t2} (\mathbf{P}_\alpha \, {}^{RC}_{a}D^{\alpha}_{b}Q - H') \, dt = 0.$$

To satisfy the variations of the following integrals, we must have

$$\frac{dF}{dt} = p_\alpha\ {}^{RC}_{a}D^{\alpha}_{b}q - \mathbf{P}_\alpha\, {}^{RC}_{a}D^{\alpha}_{b}Q + H' - H, \qquad (3.2.2)$$

where the function F is given as

$$F = S({}^{RC}_{a}D^{\alpha-1}_{b}q,\ \mathbf{P}_\alpha, t) - \mathbf{P}_\alpha\, {}^{RC}_{a}D^{\alpha-1}_{b}Q. \qquad (3.2.3)$$

Taking the total differential of F, we get

$$\frac{dF}{dt} = \frac{dS}{dt} - \frac{d\,\mathbf{P}_\alpha}{dt}\, {}^{RC}_{a}D^{\alpha-1}_{b}Q - \mathbf{P}_\alpha \frac{d}{dt}\left({}^{RC}_{a}D^{\alpha-1}_{b}Q\right)$$

$$= \frac{dS}{dt} - \frac{d\,\mathbf{P}_\alpha}{dt}\, {}^{RC}_{a}D^{\alpha-1}_{b}Q - \mathbf{P}_\alpha\ {}^{RC}_{a}D^{\alpha}_{b}Q. \qquad (3.2.4)$$

Comparing Eq:(3.2.4) to Eq. (3.2.2) we obtain

$$\frac{dS}{dt} = \frac{d\,\mathbf{P}_\alpha}{dt}\, {}^{RC}_{a}D^{\alpha-1}_{b}Q + p_\alpha\ {}^{RC}_{a}D^{\alpha}_{b}q + H' - H. \qquad (3.2.5)$$

But $S = S\left({}^{RC}_{a}D^{\alpha-1}_{b}q, P_\alpha, t\right)$. Thus, taking the total time derivative of this function, we have

$$\frac{dS}{dt} = \frac{\partial S}{\partial\, {}^{RC}_{a}D^{\alpha-1}_{b}q}\, {}^{RC}_{a}D^{\alpha}_{b}q + \frac{\partial S}{\partial\, \mathbf{P}_\alpha}\frac{d\mathbf{P}_\alpha}{dt} + \frac{\partial S}{\partial t}. \qquad (3.2.6)$$

Comparing Eq:(3.2.5) to Eq. (3.2.6), we obtain the Hamilton-Jacobi equations of motion:

$$\frac{\partial S}{\partial\, {}^{RC}_{a}D^{\alpha-1}_{b}q} = p_\alpha, \qquad \frac{\partial S}{\partial\, \mathbf{P}_\alpha} = {}^{RC}_{a}D^{\alpha-1}_{b}Q, \qquad H' = \frac{\partial S}{\partial t} + H \qquad (3.2.7)$$

If the new variables $(Q, \mathbf{P}_\alpha)$ are constants in time, then $H' = 0$ (time- independent Hamiltonian). Then we obtain the Hamilton-Jacobi partial differential equation for fractional systems as

$$\frac{\partial S}{\partial t} + H(q, p_\alpha, t) = 0. \qquad (3.2.8)$$

If P_α is constant, letting P_α =E, then the action function S $({}^{RC}_{a}D^{\alpha-1}_{b}q, E, t)$ can be put in the form

$$S= W\,({}^{RC}_{a}D^{\alpha-1}_{b}q, E) + f(E, t), \tag{3.2.9}$$

where W is called Hamilton's characteristic function, and $f\,(E, t) =-Et$.

Using Eq(3.2.7):

$$\frac{\partial S}{\partial E} = {}^{RC}_{a}D^{\alpha-1}_{b}Q = \lambda = const; \tag{3.2.10}$$

$$\frac{\partial S}{\partial\, {}^{RC}_{a}D^{\alpha-1}_{b}q} = p_\alpha = \frac{\partial W}{\partial\, {}^{RC}_{a}D^{\alpha-1}_{b}q}; \tag{3.2.11}$$

$$\frac{\partial S}{\partial t} = -H=-E. \tag{3.2.12}$$

Now, we shall find the solution of Eq: (3.2.9) to understand the physical meaning of W $({}^{RC}_{a}D^{\alpha-1}_{b}q, E)$;

$$\frac{dW}{dt} = \frac{\partial W}{\partial\, {}^{RC}_{a}D^{\alpha-1}_{b}q}\frac{d}{dt}\left({}^{RC}_{a}D^{\alpha-1}_{b}q\right)$$

$$= \frac{\partial W}{\partial\, {}^{RC}_{a}D^{\alpha-1}_{b}q}\, {}^{RC}_{a}D^{\alpha}_{b}q.$$

Using Eq. (3.2.11), we get

$$\frac{dW}{dt} = p_\alpha\, {}^{RC}_{a}D^{\alpha}_{b}q,$$

which can be written as

$$W = \int p_\alpha \, d\, {}^{RC}_{a}D^{\alpha-1}_{b} q, \quad (3.2.13)$$

which is just the Abbreviated action; defined as the integral of the generalized momentum along a path in the generalized coordinates without regard to its parameterization by time.

3.3 The Fractional Harmonic Oscillator

The Hamiltonian functions of the Fractional Harmonic Oscillator read:

$$H_f = \frac{{p_\alpha}^2}{2} + \frac{1}{2} K\, q^2.$$

Making use of Eq: (3.1.7), we obtain the corresponding fractional Hamilton's equations of motion for the fractional harmonic oscillator using RCFD, which take the form

$$\frac{\partial H_f}{\partial p} = {}^{RC}_{a}D^{\alpha}_{b} q = p_\alpha; \qquad \frac{\partial H_f}{\partial q} = -{}^{R}_{a}D^{\alpha}_{b} p_\alpha = Kq. \quad (3.3.1)$$

Then, we have

$$-{}^{R}_{a}D^{\alpha}_{b} \left({}^{RC}_{a}D^{\alpha}_{b} q\right) - Kq = 0. \quad (3.3.2)$$

Substituting for $\alpha = 1$ in Eq. (3.4.2), we obtain

$$\ddot{q} + Kq = 0.$$

Thus, the conventional result is recovered.

With Eq: (3.2.12), the fractional Hamilton-Jacobi equation (FH-JE) takes the form

$$\frac{{P_\alpha}^2}{2} + \frac{Kq^2}{2} - E = 0.$$

From Eq. (3.2.11), the above equation can be written as

$$\frac{1}{2}\left(\frac{\partial W}{\partial\, {}^{RC}_{a}D^{\alpha-1}_{b} q}\right)^2 + \frac{Kq^2}{2} - E = 0,$$

which has the following solution

$$W=\sqrt{2E - kq^2}\ {}^{RC}_{a}D^{\alpha-1}_{b} q.$$

Thus, the action function reads

$$S=\sqrt{2E - kq^2}\ {}^{RC}_{a}D^{\alpha-1}_{b} q - Et.$$

Using Eq,s. (3.2.10) and (3.2.11), we have

$${}^{RC}_{a}D^{\alpha-1}_{b} Q=\frac{1}{\sqrt{2E-kq^2}}\ {}^{RC}_{a}D^{\alpha-1}_{b} q - t = \lambda.$$

Thus, we get

$${}^{RC}_{a}D^{\alpha-1}_{b} q = \sqrt{2E - kq^2}(t + \lambda).$$

Taking the total time derivative of both sides, we have

$${}^{RC}_{a}D^{\alpha}_{b} q = \sqrt{2E - kq^2} = p_{\alpha}.$$

Taking again the Riesz derivative of both sides and making use of Eq (3.3.1), we obtain

$$ {}_{a}^{R}D_{b}^{\alpha}\left({}_{a}^{RC}D_{b}^{\alpha}q\right) = {}_{a}^{R}D_{b}^{\alpha}\, p_{\alpha} = -kq; $$

$$ {}_{a}^{R}D_{b}^{\alpha}\left({}_{a}^{RC}D_{b}^{\alpha}q\right) + Kq = 0. $$

This is the same result as that obtained from the fractional Hamilton's equation of motion Eq: (3.3.2).

Chapter Four

Path Integral Quantization

In this chapter, the path integral method for quantization will be outlined using the classical action function. Then the method will be applied to the fractional Hamiltonian of a harmonic oscillator with Riesz Caputo fractional derivatives.

4.1 Introduction to the Path Integral Formulation

The basic difference between the classical and quantum mechanics is that in classical mechanic only a definite path contributes to the motion of the system; while in quantum mechanics all possible paths must play a role in the motion of the system (Feynman postulate 2):

"The paths contribute equally in magnitude, but the phase of their contribution is the classical action S in units of the quantum of action ħ" [Feynman, 1965].

$$\varphi[q(t)] = \text{const.}\, e^{\frac{i}{\hbar} s[q(t)]},$$

where the classical action S is defined by:

$$S[q(t)] = \int_{t_a}^{t_b} [L(q, \dot{q}, t)\, dt]. \tag{4.1}$$

The total amplitude in going from q_a to q_b is the sum of $\varphi[q(t)]$ over all paths

$$K(q_a, q_b) = A(t) \sum_{\text{all paths}} \varphi[q(t)], \tag{4.2}$$

where A is a normalization factor whose value is independent of any individual path but depends only on time. For a Lagrangian of the form

$$L= m\frac{\dot{q}^2}{2} - V (q, t),$$

The normalization factor is [Feynman, 1965]

$$A =\left(\frac{2\pi i\hbar \epsilon}{m} \right)^{\frac{1}{2}}; \qquad \epsilon = \frac{t_f - t_i}{N}.$$

To perform the sum over all paths in Eq. (4.2), we divide the time interval $(t_b - t_a)$ into N intervals of length ϵ. In the limit $\epsilon \rightarrow 0$, the sum becomes a multiple integral over all values of q_i:

$$K (b, a) = \lim_{\epsilon \rightarrow 0} \frac{1}{A} \iint \ldots \int e^{\frac{i}{\hbar}S[b,a]} \frac{dq_1}{A} \frac{dq_2}{A} \ldots \frac{dq_{N-1}}{A}$$

$$K (b, a) = \int_a^b e^{\frac{i}{\hbar}s[q(t)]} Dq(t),$$

where $Dq(t)$ stands for $dq_1\, dq_2 \ldots dq_{N-1}$, and $K(b, a)$ is called the propagator or the kernel of the motion. In fact, $K (q_b, t_b, q_a, t_a)$ is a wave function; $\Psi (q_b, t_b)$.

The phase space- path integral is given by

$$K (q_f, t_f, q_i, t_i) = \int Dq\, Dp_j\, e^{\frac{i}{\hbar} \int \left(p_j \dot{q}_j - H(p,q)\right) dt}, \qquad (4.3)$$

where

$$Dq = \prod_{j=1}^{n-1} dq_j \quad \text{and} \quad Dp_j = \prod_{j=0}^{n-1} \frac{dp_j}{2\pi\hbar}.$$

4.2 Quantization of the Fractional Harmonic Oscillator

The Hamiltonian and the path integral quantization of a system with Caputo fractional derivatives were developed by [Baleanu, Muslih and Rabei 2006]. A general formula for the path-integral quantization for non-conservative systems with RLFD and CFD was constructed by [Tarawneh, K. 2008].

In this section, the quantization of the fractional harmonic oscillator using Riesz-Caputo fractional derivatives (RCFD) will be carried out according to the path- integral method.

For a fractional harmonic oscillator, the Hamiltonian reads

$$H_f = \frac{{p_\alpha}^2}{2} + \frac{1}{2} K\, q^2.$$

With Eq. (4.2), the phase-space path integral of the fractional harmonic oscillator can be written as

$$K\,[q(t)] = \int dq\; dp_\alpha \;\; e^{\frac{i}{\hbar}\int\left(p_\alpha\; {}^{RC}_{\ a}D^{\alpha}_{b}q - \frac{{P_\alpha}^2}{2} - \frac{kq^2}{2}\right)dt}.$$

Integrating over $p_{\alpha,}$ using a Gaussian integral, and substitute k $=m\omega^2$, we obtain the space path integral

$$K = \int dq \;\; e^{\frac{i}{\hbar}\int\left(\frac{({}^{RC}_{\ a}D^{\alpha}_{b}q)^2}{2} - \frac{m\omega^2 q^2}{2}\right)dt}.$$

Where integral in the exponent is nothing but the classical action function.

Accordingly, the probability of going from point q_a at time t_a to q_b at t_b is

$$P\,(b, a) = \left| K(q_b, t_b, q_a, t_a) \right|^2.$$

Chapter Five

A Lagrangian Formulation for a Damped Harmonic Oscillator Using RFD

The construction of the Lagrangian and the Hamiltonian for the damped harmonic oscillator (DHO) is problematic because it leads to an explicitly time-dependent Lagrangian [Dekker 1981].

Riewe [Riewe, 1996 and 1997] tried to propose a Lagrangian for a damped harmonic oscillator using RLFD; but he committed a technical error; where, he considered the left RLFD is equal to the right RLFD.

Recently, the Lagrangian and Hamiltonian formulations for DHO have been presented using CFD by [Tarawneh, K. 2008]. In his work, he used the symmetric fractional derivative for CFD presented in [Klimek 2001].

By taking the limit $b \to a^+$, he considered that the LCFD and the RCFD of order $(\frac{1}{2})$ are related as

$$i\,{}_a^C D_t^{\frac{1}{2}}\, q = ({}_t^C D_b^{\frac{1}{2}}\, q),$$

and from this consideration he built his Lagrangian for DHO.

In this chapter, we discuss the construction of the Lagrangian for a damped harmonic oscillator using Riesz fractional derivatives.

Consider a Lagrangian of the form $L\,(q,\ {}^{R}_{a}D^{\alpha}_{b}q,\ \dot{q}\,)$.

One can show that the corresponding Euler-Lagrangian equation of motion reads

$$\frac{\partial L_f}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_f}{\partial \dot{q}}\right) - {}^{RC}_{a}D^{\alpha}_{b}\left(\frac{\partial L_f}{\partial\, {}^{R}_{a}D^{\alpha}_{b}q}\right) = 0. \qquad (5.1)$$

Now, we propose the following Lagrangian

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\left({}^{R}_{a}D^{\alpha}_{b}q\right)^2 - \frac{1}{2}\,k\,q^2.$$

This gives the equation of motion for the Damped Harmonic Oscillator. Making use of Eq. (5.1), we obtain

$$\ddot{q} + {}^{RC}_{a}D^{\frac{1}{2}}_{b}\left({}^{R}_{a}D^{\frac{1}{2}}_{b}q\right) + Kq = 0. \qquad (5.2)$$

For $0 < \alpha < 1$ Eq,s. (2.13) and (2.14) become

$${}^{R}_{a}D^{\alpha}_{b}q(t) = \frac{1}{2}\left({}_{a}D^{\alpha}_{t}\,q(t) - {}_{t}D^{\alpha}_{b}\,q(t)\right); \qquad (5.3)$$

and

$${}^{RC}_{a}D^{\alpha}_{b}q(t) = \frac{1}{2}\left({}^{C}_{a}D^{\alpha}_{t}\,q(t) - {}^{C}_{t}D^{\alpha}_{b}\,q(t)\right). \qquad (5.4)$$

Using the relation between the Riemann-Liouville Fractional Derivatives and Caputo Fractional Derivatives for $0 < \alpha < 1,$ we have

$$ {}_{a}^{C}D_{t}^{\alpha}\, q(t) = {}_{a}D_{t}^{\alpha}\, q(t) - \frac{(t-a)^{-\alpha} q(a)}{\Gamma(1-\alpha)}; \qquad (5.5) $$

$$ {}_{t}^{C}D_{b}^{\alpha}\, q(t) = {}_{t}D_{b}^{\alpha}\, q(t) - \frac{(b-t)^{-\alpha} q(b)}{\Gamma(1-\alpha)}. \qquad (5.6) $$

For $\alpha = \frac{1}{2}$, both Eq,s (5.5) and (5.6) become

$$ {}_{a}^{C}D_{t}^{\frac{1}{2}}\, q = {}_{a}D_{t}^{\frac{1}{2}}\, q(t) - \frac{(t-a)^{-\frac{1}{2}} q(a)}{\Gamma(\frac{1}{2})}; \qquad (5.7) $$

$$ {}_{t}^{C}D_{b}^{\frac{1}{2}}\, q = {}_{t}D_{b}^{\frac{1}{2}}\, q(t) - \frac{(b-t)^{-\frac{1}{2}} q(b)}{\Gamma(\frac{1}{2})}. \qquad (5.8) $$

Substituting Eqs. (5.7) and Eq. (5.8) in Eq. (5.3), we obtain

$$ {}_{a}^{R}D_{b}^{\frac{1}{2}} q = \frac{1}{2}\left(({}_{a}^{C}D_{t}^{\frac{1}{2}}\, q + \frac{(t-a)^{-\frac{1}{2}} q(a)}{\sqrt{\pi}} - {}_{t}^{C}D_{b}^{\frac{1}{2}}\, q - \frac{(b-t)^{-\frac{1}{2}} q(b)}{\sqrt{\pi}} \right), $$

$$ = \left(\frac{1}{2}\left({}_{a}^{C}D_{t}^{\frac{1}{2}}\, q - {}_{t}^{C}D_{b}^{\frac{1}{2}}\, q \right) + \frac{1}{2}\, (\frac{(t-a)^{-\frac{1}{2}} q(a)}{\sqrt{\pi}} - \frac{(b-t)^{-\frac{1}{2}} q(b)}{\sqrt{\pi}}) \right). \qquad (5.9) $$

With Eq. (5.4) and the boundary conditions $q(a) = q(b) = 0$,

Eq. (5.9) becomes

$$ {}_{a}^{R}D_{b}^{\frac{1}{2}} q = {}_{a}^{RC}D_{b}^{\frac{1}{2}}\, q. \qquad (5.10) $$

Substituting Eq (5.10) in (5.2);

$$\ddot{q} + {}^{RC}_{a}D^{\frac{1}{2}}_{b}\left({}^{RC}_{a}D^{\frac{1}{2}}_{b}q\right) + Kq = 0;$$

$$\ddot{q} + \dot{q} + Kq = 0, \qquad (5.11)$$

which is the equation of motion for a damped harmonic oscillator.

Chapter Six

Conclusion

6.1 General Summary

During the last decades fractional calculus has become an alternative tool for solving several complex problems in various fields. The fractional derivative represents an operator which generalizes the ordinary derivative.

In this work, we have presented definitions of the Riesz fractional integral (potential) and derivatives, and their properties. We have also defined the Riemann–Liouville and Caputo derivatives; as they are linked to Riesz fractional derivatives. Generalized Euler–Lagrange equations have been presented for fractional variational problems FVP defined in terms of Riesz and Riesz Caputo fractional derivatives. We note that both the RFD and the RCFD automatically appear in the resulting differential equations even when the functional contains only one of them (RCFD or RFD) [Baleanu 2007].

The Hamilton–Jacobi equation for systems containing RCFD has been constructed using the canonical method. The Hamilton–Jacobi functions have then been obtained by solving these equations. Finding this function enables us to get the solutions of the equations of motion. In order to test our formalism, and to get a somewhat deeper understanding, the fractional harmonic oscillator with Riesz- Caputo fractional derivatives is discussed. The result was found to be in exact agreement with the Lagrangian formulation given by [Agrawal 2002] and with the Hamiltonian formulation given by [Rabei et al. 2007].

The advantage of using the method presented in this thesis is that we can easily obtain the action function, which is an essential part in the path integral quantization for any mechanical fractional system.

A new formulation of the Lagrangian is derived for a damped harmonic oscillator using Riesz fractional derivatives, and the corresponding equation of motion is obtained.

6.2 Open Problems

The first problem is to obtain a general formula for fractional Poisson brackets using RFD and then perform the canonical quantization by using the commutation relation between momentum and space in the fractional form.

The second one is to discuss the quantization of the Damped Harmonic Oscillator.

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التكمية للمتذبذب التوافقي الكسري بدلالة مشتقة ريز الكسريّة

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ملخص

في هذه الأطروحة، أعيد تعريف مشتقتي ريمان- لويڤل و كابوتو بما يرتبط مع المشتقة الكسرية لريز.

وأعيدت صياغة اللغرانجيّة والهاملتونيّة الكسريين في الأنظمة التي تحتوي على المشتقة الكسرية لريز. كما عرّفت معادلات هاملتون في الحركة بالمشتقة الكسرية نفسها، بالإضافة إلى ذلك ، أنشئت معادلة هاملتون- جاكوبي للأنظمة المحافظة التي تحتوي مشتقة ريز الكسرية .

شرح مثال توضيحي لهذه المعادلات، وطبقت هذه المعادلات على المتذبذب التوافقي البسيط. واستردت النتائج الكلاسيكيّة لمعادلات الحركة.

جرى تكمية المذبذب التوافقي البسيط باستخدام المشتقة الكسرية لريز و ريز كابوتو وذلك بطريقة تكامل المسار.

وأخيرا استطعنا الحصول على الصيغة اللاغرانجيّة المكافئة للنظام الذي يحوي قوة مبددّة. وأوجدنا معادلة الحركة لهذا النظام باستخدام مشتقة ريز الكسريّة.